Christ Church Grammar School

2019
TEST 4

MATHEMATICS SPECIALIST Year 12

Section One:
Calculator-free

Your name SOLUTIONS

Teacher's name _____

Time and marks available for this section

Reading time for this section:	2 minutes
Working time for this section:	15 minutes
Marks available:	16 marks

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Instructions to candidates

1. The rules of conduct of the CCGS assessments are detailed in the Reporting and Assessment Policy. Sitting this assessment implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. Answer all questions.

4. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

5. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

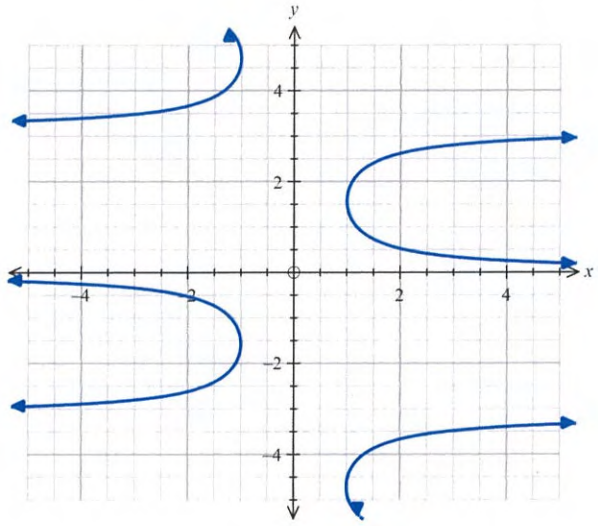
6. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

7. It is recommended that **you do not use pencil**, except in diagrams.

See next page

Question 1 **(4 marks)**

(a) Find the expression for $\frac{dy}{dx}$ given the relationship $x \sin y = 1$ shown graphed below. (3 marks)



$1.\sin y + x\cos y \cdot \frac{dy}{dx} = 0$ ✓ implicitly differentiate using product rule

$x\cos y \frac{dy}{dx} = -\sin y$

$\frac{dy}{dx} = -\frac{\sin y}{x\cos y}$ ✓ re-arrange

$= -\frac{\tan y}{x}$ ✓ simplifies

(b) Hence find the coordinates of the point on the curve in quadrant one where $\frac{dy}{dx}$ is undefined. (1 mark)

$\frac{dy}{dx}$ is undefined where $\tan y$ is not defined

ie at $y = \frac{\pi}{2}$

If $y = \frac{\pi}{2}$, $x\sin\left(\frac{\pi}{2}\right) = 1$

$x = 1$

$\therefore$ The point is $\left(1, \frac{\pi}{2}\right)$ ✓

no working required

See next page

Question 2 **(12 marks)**

(a) Evaluate the following definite integrals, giving your answers in exact form.

(i) $\int_1^e \frac{(\ln x)^2}{x}\,dx$ let $u = \ln x$ (3 marks)

$= \int_0^1 \frac{u^2}{x} \cdot x\,du$ ✓

$= \int_0^1 u^2\,du$

$= \frac{u^3}{3}\Big]_0^1$ ✓

$= \frac{1^3}{3} - \frac{0^3}{3}$

$= \frac{1}{3}$ ✓

$u = \ln x$

$\frac{dy}{dx} = \frac{1}{x}$

$dx = x\,du$

If $x = e$, $u = 1$

$x = 1$, $u = 0$

(ii) $\int_0^{\frac{\pi}{12}} \sin^3(3x)\cos(3x)\,dx$ (4 marks)

not needed

let $u = \sin 3x$

$= \int_0^{\frac{1}{\sqrt{2}}} u^3 \cos 3x \frac{du}{3\cos 3x}$ ✓ correct subst or $\int_0^{\pi/12} \sin^3(3x).\cos 3x\,dx$ $\frac{du}{dx} = 3\cos 3x$

correct ∫ and simplify

$= \int_0^{\frac{1}{\sqrt{2}}} \frac{u^3}{3}\,du$ ✓ correct $\int du$ or $= \left[\frac{\sin^4(3x)}{12}\right]_0^{\pi/12}$ ✓ $\frac{du}{3\cos 3x} = dx$

$= \frac{u^4}{12}\Big]_0^{\frac{1}{\sqrt{2}}}$ ✓ correct evaluation of ∫ or $= \frac{\left(\sin\left(\frac{\pi}{4}\right)\right)^4}{12} - \frac{(\sin 0)^4}{12}$ ✓ correct subst (or evaluation)

If $x = \frac{\pi}{12}$ $u = \frac{1}{\sqrt{2}}$

If $x = 0$ $u = 0$

$= \frac{\left(\frac{1}{\sqrt{2}}\right)^4}{12} - 0$ $= \frac{\left(\frac{1}{\sqrt{2}}\right)^4}{12} - 0$

$= \frac{1}{48}$ ✓ correct value $= \frac{1}{48}$. ✓ correct value

See next page

Question 2 continued

(b) $\int \frac{1+\tan x}{1-\tan x}\,dx$ (5 marks)

Hint: Consider $\tan(x+y) = \frac{\tan x+\tan y}{1+\tan x \tan y}$

$= \int \frac{\tan \pi/4 + \tan x}{1 - \tan\frac{\pi}{4}\tan x}\,dx$ ✓ express in this form

$= \int \tan\left(\frac{\pi}{4} + x\right) dx$ ✓ simplifies to tan( ) rule

$= \int \frac{\sin\left(\frac{\pi}{4}+x\right)}{\cos\left(\frac{\pi}{4}+x\right)}\,dx$ ✓ changes to sin & cos.

$u = \cos\left(\frac{\pi}{4}+x\right)$

$\frac{du}{dx} = -\sin\left(\frac{\pi}{4}+x\right)$

$dx = \frac{du}{-\sin\left(\frac{\pi}{4}+x\right)}$

$= \int \frac{\sin\left(\frac{\pi}{4}+x\right)}{u} \cdot \frac{du}{-\sin\left(\frac{\pi}{4}+x\right)}$

$= -\int \frac{1}{u}\,du$ ✓ correct ∫

$= -\ln|u| + C$

$= -\ln\left|\cos\left(\frac{\pi}{4}+x\right)\right| + C$ ✓ correct solution with 'x' subst back in (if did subst).

End of questions

Additional working space

Question number: _____

Additional working space

Question number: _____

Christ Church Grammar School

2019
TEST 4

MATHEMATICS SPECIALIST Year 12

Section Two:
Calculator-assumed

Your name SOLUTIONS

Teacher's name _____

Time and marks available for this section

Reading time for this section:	3 minutes
Working time for this section:	30 minutes
Marks available:	25 marks

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates and up to three calculators approved for use in the WACE examinations

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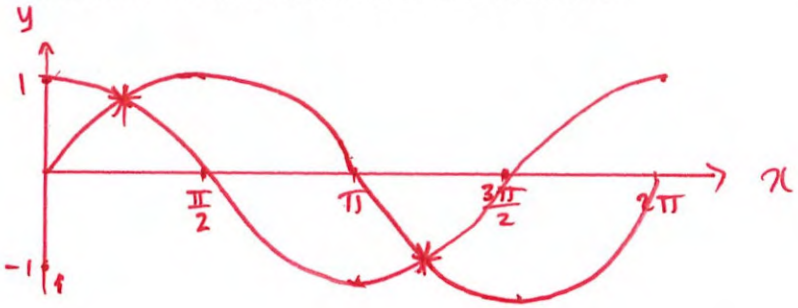
See next page

Question 3 **(7 marks)**

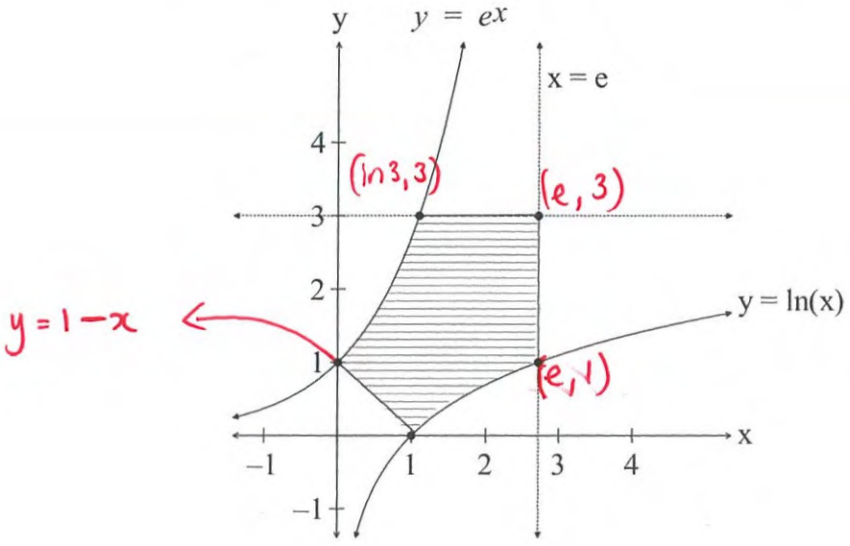
(a) Determine the area between $f(x) = \sin(x)$ and $g(x) = \cos(x)$ between two consecutive intersections of the functions. (3 marks)



$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx = 2.83 \text{ units}^2$$

correct limits ✓ correct subtraction ✓ correct value. Accept any rounding ✓

(b) Find the expression that if evaluated represents the shaded area in the diagram below. (Do not evaluate the expression.) (4 marks)



$$A = \int_0^{\ln 3} e^x\,dx + \int_{\ln 3}^{e} 3\,dx - \int_0^1 (1-x)\,dx - \int_1^e \ln(x)\,dx$$

✓ ✓ ✓ ✓

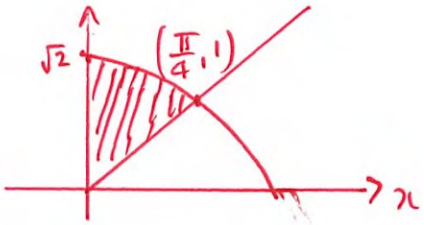
or $3e - \left(\frac{1}{2} + \int_1^e \ln x\,dx + \int_1^3 \ln y\,dy\right)$

See next page

Question 4 **(5 marks)**

The curve $y = \sqrt{2}\cos(x)$ and the line $y = \frac{4}{\pi}x$ intersect at the point $\left(\frac{\pi}{4}, 1\right)$.

(a) Determine the area bound by the curve, the line and the y-axis. (3 marks)



$$A = \int_0^{\frac{\pi}{4}} \left(\sqrt{2}\cos x - \frac{4x}{\pi}\right) dx.$$

✓ correct limits ✓ correct ∫

$$= 1 - \frac{\pi}{8}$$ ✓ correct answer

or 0.6073 (4dp) accept any rounding.

(b) Calculate the volume formed when the area is rotated about the x-axis. (2 marks)

$$V = \pi \int y^2 \, dx$$

$$V = \pi \int_0^{\pi/4} (\sqrt{2}\cos x)^2 dx - \pi \int_0^{\pi/4} \left(\frac{4x}{\pi}\right)^2 dx$$ ✓ correct set up of volume

$$= \pi\left(\frac{\pi}{6} + \frac{1}{2}\right) \text{ units}^3$$ ✓ correct solution

or $\frac{\pi^2}{6} + \frac{\pi}{2}$

or $\frac{\pi(\pi+3)}{6}$ or 3.21573 units^3

accept any rounding.

See next page

Question 5 **(4 marks)**

Show, using partial fractions and Calculus techniques, that

$$\int_2^4 \frac{x+3}{x(x-1)}\,dx = \ln\frac{81}{8}$$

$\frac{x+3}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$

$\Rightarrow x+3 = A(x-1) + Bx$

If $x=0$: $3 = -A \therefore A = -3$

If $x=1$: $4 = B$ ✓ finds values for partial fractions

$\therefore \int_2^4 \left(-\frac{3}{x} + \frac{4}{x-1}\right) dx$

$= -3\ln|x| + 4\ln|x-1| \Big]_2^4$ ✓ correct $\int$

$= -3\ln 4 + 4\ln 3 - (-3\ln 2 + 4\ln 1)$ ✓ correct subst

$= 4\ln 3 - 3\ln 4 + 3\ln 2 - 4\ln 1$

$= \ln 3^4 - \ln 4^3 + \ln 2^3 - 0$

$= \ln\left(\frac{3^4 \times 2^3}{4^3}\right)$

$= \ln\left(\frac{81 \times 8}{64}\right)$ ✓ correct simplification.

$= \ln\left(\frac{81}{8}\right)$

See next page

Question 6 **(4 marks)**

On a suitable domain, a curve is defined parametrically by

$$x = t^2 + 1 \text{ and } y = \ln(3t + 2).$$

Find the equation of the normal to the curve where $t = -\frac{1}{3}$.

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{3}{3t+2} \times \frac{1}{2t}$

$\frac{dx}{dt} = 2t \quad \therefore \frac{dt}{dx} = \frac{1}{2t}$

$\frac{dy}{dx} = \frac{3}{3t+2}$

$\frac{dy}{dx} = \frac{3}{2t(3t+2)}$ ✓ Finds $\frac{dy}{dx}$ correctly

when $t = -\frac{1}{3}$ $\frac{dy}{dx} = \frac{3}{2(-\frac{1}{3})(3(-\frac{1}{3})+2)}$

$= -\frac{9}{2}$

$\therefore$ m of $\perp$ $= \frac{2}{9}$ ✓ Finds $\perp$ of m

If $t = -\frac{1}{3}$, $x = \frac{10}{9}$, $y = 0$ or $(\frac{10}{9}, 0)$ ✓ Finds 'x' & 'y' value

$\therefore$ Eqn of normal $y = \frac{2}{9}x + c$

$0 = \frac{2}{9} \times \frac{10}{9} + c$

$c = -\frac{20}{81}$

$\therefore y = \frac{2}{9}x - \frac{20}{81}$ or $81y = 18x - 20$ ✓ Finds eqn of normal.

Note: can use Classpad to find equation but must show sufficient working for 4 marks.

See next page

Question 7 **(5 marks)**

Use the substitution $u = 5x^2$ and then $u = \sin\theta$, to find the exact value of

$$\int_0^{\frac{1}{\sqrt{10}}} \frac{x}{\sqrt{1-25x^4}}\,dx.$$

$$= \int_0^{\frac{1}{\sqrt{10}}} \frac{x}{\sqrt{1-(5x^2)^2}}\,dx$$

$u = 5x^2$

$\frac{du}{dx} = 10x$

$dx = \frac{du}{10x}$

$x = 0,\ u = 0$

$x = \frac{1}{\sqrt{10}}\quad u = 5\left(\frac{1}{\sqrt{10}}\right)^2 = \frac{1}{2}$

$$= \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-u^2}}\,\frac{du}{10x}$$ ✓ uses subst to find $\int$

$$= \frac{1}{10}\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-u^2}}\,du$$

now let $u = \sin\theta$

$\frac{du}{d\theta} = \cos\theta$ ✓ chooses 2nd subst.

$du = \cos\theta\,d\theta$

If $u = 0\quad \theta = 0$

$u = \frac{1}{2}\quad \theta = \frac{\pi}{6}$

$$= \frac{1}{10}\int_0^{\pi/6} \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta$$ ✓ sets up $\int$ correctly

$$= \frac{1}{10}\int_0^{\pi/6} \frac{1}{\sqrt{\cos^2\theta}}\cos\theta\,d\theta$$

$$= \frac{1}{10}\int_0^{\pi/6} \frac{1}{\cos\theta}\cdot\cos\theta\,d\theta$$

$$= \frac{1}{10}\int_0^{\pi/6} 1\,d\theta$$ ✓ simplifies $\int$

$$= \frac{1}{10}\left[\theta\right]_0^{\pi/6}$$

$$= \frac{1}{10}\left[\frac{\pi}{6} - 0\right]$$

$$= \frac{\pi}{60}$$ ✓ correct answer.

End of Questions

Additional working space

Question number: _____

Additional working space

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